Course review.

Monday, December 4, 2023 12:00 PM

Let $\Omega \subset \mathbb{C}$ be a simply-connected region

The following are equivalent definitions of an analytic function in Ω:

1) $\forall z \in \Omega$: $\exists \lim\limits_{w \to z} \frac{f(w)-f(z)}{w-z} = f'(z)$.

2) f is real-differentiable in Ω and $\frac{\partial f}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y}\right) = 0$ (Cauchy-Riemann)

in real form : $f = u + iv$: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

3) f is real-differentiable in Ω and $\forall z$:

$f(z+h) = f(z) + T_z(h) + o(h)$, where T_z is complex-linear.

4) $\forall \gamma \subset \Omega$ - closed, $\oint\limits_\gamma f(z)\,dz = 0$. (Cauchy Theorem) $\gamma \sim 0$ in General.

4sc) $\exists F$ - analytic in Ω, $f = F'$.

5) $\forall z \in \Omega$ $\exists r > 0$: $\forall R \subset B(z,r)$ - rectangle with sides parallel to the axes, $\oint\limits_{\partial R} f(z)\,dz = 0$.

6) $\forall \gamma$ - closed: $f(z)\, n(\gamma, z) = \frac{1}{2\pi i} \oint\limits_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta$. (Cauchy integral formula) $\gamma \sim 0$ in General
$z \in \Omega$

7) $f \in C^\infty(\Omega)$ + Cauchy-Riemann

8) $\forall z_0 \in \Omega$, $\exists r > 0$: $|z - z_0| < r \Rightarrow f(z) = \sum\limits_{n=0}^{\infty} a_n (z - z_0)^n$.

9) $\forall z_0 \in \Omega$ $|z - z_0| < \operatorname{dist}(z_0, \partial\Omega) \Rightarrow f(z) = \sum\limits_{n=0}^{\infty} a_n (z - z_0)^n$,

$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \oint\limits_{C_r} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\,d\zeta, \text{ where}$$

$r < \operatorname{dist}(z_0, \partial\Omega)$, $C_r = \{z_0 + re^{it},\ 0 \le t \le 2\pi\}$.

If Ω is an arbitrary region then 4sc) does not hold, but 1)-3), 5), 7)-9) still hold, 4), 6) hold for $\gamma \sim 0$.

For 4sc) - take any $\gamma \not\sim 0$ in Ω, $z_0 \notin \Omega$: $n(\gamma, z_0) \neq 0$. Then $\frac{1}{z - z_0}$ does not have antiderivative.

General Cauchy Integral Formula:

Let γ be a cycle, $\gamma \sim 0$ in Ω. $f \in A(\Omega)$.

Then if $z \notin \gamma$, $z \in \Omega$, then

$n(\gamma, z)\, f^{(k)}(z) = \frac{k!}{\ldots} \oint \frac{f(\zeta)}{\ldots}\,d\zeta$

Then if $z \notin \gamma$, $z \in \Omega$, then

$$n(\gamma, z) f^{(k)}(z) = \frac{k!}{2\pi i} \oint_\gamma \frac{f(\zeta)}{(\zeta - z)^{k+1}} d\zeta$$

General Argument Principle.

If $\gamma \sim 0$ in Ω, $f \in \mathcal{M}(\Omega)$, $f \neq 0, \infty$ on γ, then

$$n(f(\gamma), 0) = \frac{1}{2\pi i} \oint_\gamma \frac{f'(\zeta)}{f(\zeta)} d\zeta = \sum_{z \in \Omega} n(\gamma, z) \operatorname{ord}(f, z).$$

Complex numbers:

$z = x + iy = |z| e^{i \arg z}$, $\arg z = \{\operatorname{Arg} z + 2\pi k, k \in \mathbb{Z}\}$

Multiplication by z: rotation by $\arg z$, dilation by $|z|$.

$\hat{\mathbb{C}}$: $\mathbb{C} \cup \{\infty\}$, $d(z, z') = \frac{2|z - z'|}{\sqrt{1+|z|^2}\sqrt{1+|z'|^2}}$ $\quad d(z, \infty) = \frac{2}{\sqrt{1+|z|^2}}$

$e^z = e^x \cdot e^{iy} = e^x(\cos y + i \sin y)$ $\quad |e^z| = e^x$, $\arg e^z = \{y + 2\pi k, k \in \mathbb{Z}\}$

$\boxed{e^{z + 2\pi i} = e^z}$

$z \neq 0$ $\log z = \log|z| + i \arg z$

$\operatorname{Log} z = \log|z| + i \operatorname{Arg} z$ — not continuous on $\mathbb{R}_- = (-\infty, 0]$!

Example. $\operatorname{Log}(e^{3+5i}) \neq 3 + 5i$
$= 3 + (5 - 2\pi) i$

$\cos z = \frac{e^{iz} + e^{-iz}}{2}$ $\quad \cosh z = \frac{e^z + e^{-z}}{2}$

$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$ $\quad \sinh z = \frac{e^z - e^{-z}}{2}$

Fractional Linear (Möbius) transformations.

$f(z) = \frac{az + b}{cz + d} \in \mathcal{M}(\hat{\mathbb{C}})$, rational of degree 1.

Normalization: $ad - bc = 1$.

Preserve:
1) lines and circles
2) Symmetries wrt lines and circles.
3) Cross ratio.

Example: find all conformal maps from $\{\operatorname{Re} z > 0\}$ onto $\mathbb{D}$.

Know: need to find one, the rest - using self-map of $\mathbb{D}$. So if we find one Möbius bijection the rest will be Möbius, as compositions of Möbius maps.

$\varphi: \{\mathrm{Re}\, z>0\} \to \mathbb{D}$ - Möbius.

Let $\varphi(z_0)=0$, then $\varphi(-\bar{z}_0)=\infty$.

So $\varphi(z)=c\,\dfrac{z-z_0}{z+\bar{z}_0}$ for some c.

$\partial\mathbb{D} \ni \varphi(0)=c\,\dfrac{z_0}{\bar{z}_0} \Rightarrow |c|=1.$

So $\varphi(z)=e^{i\vartheta}\,\dfrac{z-z_0}{z+\bar{z}_0}$, $\mathrm{Re}\, z_0>0$ - general form.

If $z=it\ (t\in\mathbb{R})$, $|\varphi(z)|=\left|\dfrac{it-z_0}{it+\bar{z}_0}\right|=\left|\dfrac{it-z_0}{-\overline{(it-z_0)}}\right|=1.$

So $\varphi: i\mathbb{R}\to\partial\mathbb{D}$, $\varphi(z_0)=0$ - indeed maps $\{\mathrm{Re}\, z>0\}$ onto $\mathbb{D}$.

Locally uniform convergence:

Let (f_n) be a sequence of functions on a region Ω.

TFAE:

1) $\forall K\subset\Omega$ - compact, $f_n \rightrightarrows f$ uniformly on K.

2) $\forall z\in\Omega\ \forall\varepsilon>0\ \exists\,\delta(\varepsilon,z), N(\varepsilon,z)$: $\left.\begin{cases}|w-z|<\delta\\ w\in\Omega\\ n>N\end{cases}\right\} \Rightarrow |f(w)-f_n(w)|<\varepsilon.$

$f_n\in A(\Omega)$, $f_n \xrightarrow{l.u.} f \Rightarrow f\in A(\Omega)$ — Weierstrass

$f_n^{(k)}\to f^{(k)}$ ✓

$f_n\neq 0 \Rightarrow f\equiv 0$ or $f\neq 0$ - Hurwitz

f_n - conformal $\Rightarrow$ f - conformal or $f\equiv$ const. - Hurwitz.

Maximum principle:

$f\in A(\Omega) \Rightarrow$ no $z_0\in\Omega$ is a local maximum of $|f(z_0)|$

if $f(z_0)\neq 0$, - not a local minimum (consider $\frac{1}{f(z)}$).

Uniqueness Theorem: $f,g\in A(\Omega)$,

$A = \{z : f(z) = g(z)\}$

A has a limit point in $\Omega \Rightarrow f \equiv g$.